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NATURAL FREQUENCIES OF BONDED AND UNBONDED PROPELLANTS IN THREE-AND FIVE-INCH ROCKET MOTORS

by

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The results of an analysis by Armenàkas for the natural frequencies of composite (two material) cylinders is used to determine the lowest natural frequency of bonded and unbonded propellants in three-inch and five-inch rocket motors. The lowest frequency is found to be approximately eight times larger than the lowest frequency in the launch force pulse. Thus, the dynamic effects in the propellant will be significant.

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INTRODUCTION

This analysis considers one aspect of the dynamic response of a rocket motor during the launch phase. It is part of a continuing effort devoted to the development of analytical tools for the design of gun-launched rockets. Previous studies (Refs. 1 and 2) of the rocket dynamic response have concentrated on the response of the steel motor case alone; i.e. the effects of the propellant have been neglected, corresponding to an unbonded propellant. The results of the previous studies indicate that for breech pressure loadings of the form $P_o \sin \frac{\pi t}{\alpha}$, where $0 \leq t \leq \alpha$, the significance of the stress waves, or elastic effects, in the steel case is essentially dependent upon the ratio of the natural frequencies of the case to the forcing frequency $\frac{\pi}{\alpha}$*. For steel motor cases treated as a one-dimensional bar the lowest natural frequency for axial motion was found to be considerably higher than the forcing frequency for typical loading pulses. As a consequence, the case reacts to the breech pressure as a "slowly" applied force and the stress wave effects can be neglected when determining the internal axial force in the case during launch. Hence, the internal axial force in the case can be computed on the basis of the mass distribution along the rocket and the rigid body acceleration of the rocket.

The question arises as to whether or not the same situation exists for the propellant, i.e. is the lowest natural frequency of the propellant considerably higher than the forcing frequency $\frac{\pi}{\alpha}$? If so, conventional

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*This is not a problem of resonance since $0 \leq t \leq \alpha$. Instead, it has to do with the "rise" time of the applied load.

static analysis techniques can be used to determine the stress state of both the propellant and the motor case. This question is answered here by determining the natural frequencies of the case-propellant system. The propellant is assumed to be incompressible and elastic, and three different propellant-case interface conditions are considered: (1) the propellant is bonded to the case; (2) the propellant is unbonded, but restrained from expanding radially; and (3) the propellant is free. The natural frequencies are determined using the results of an analysis by Armenàkas (Ref. 3) for the natural frequencies of composite (two material) circular cylinders. A computer program has been written and the lowest natural frequencies computed for several typical rockets. These frequencies are compared with the forcing frequency, and the significance of the dynamic elastic effects in the propellant is assessed.

EQUATIONS OF MOTION

The propellant-motor case geometry is idealized as shown below

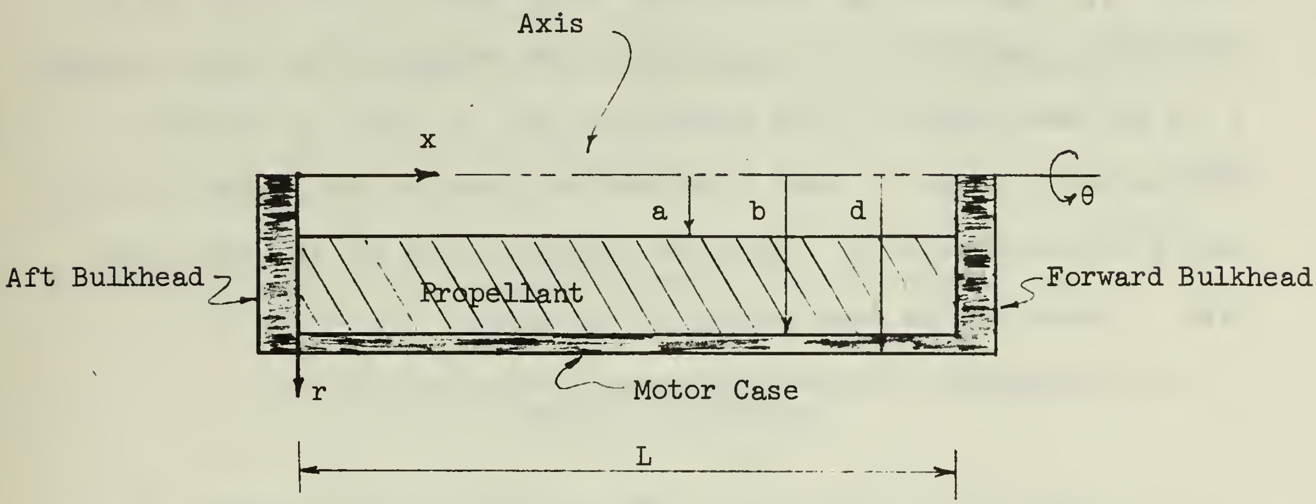

Fig. 1 Motor Case and Propellant

The cylindrical coordinate system r, θ, x is attached to the aft bulkhead of the case as shown.

Assume that the motor is subjected to an external pressure at the aft bulkhead. This pressure causes the motor, and hence the coordinate system, to translate in the direction of the x axis. Denote the motion of the origin of the coordinate system during launch by $\bar{U}(t)$, where t is time. Accordingly, the equilibrium equations of the propellant for motion symmetric about the x axis can be given in the form

$$\frac{\partial(r\sigma_{rr})}{r\partial r} + \frac{\partial\sigma_{rx}}{\partial x} - \frac{\sigma_{\theta\theta}}{r} = \rho\,\frac{\partial^2 u_r}{\partial t^2} \qquad (1a)$$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial (r\sigma_{rx})}{r\partial r} = \rho \left(\frac{\partial^2 u_x}{\partial t^2} + \frac{\partial^2 \overline{U}}{\partial t^2}\right) \tag{1b}$$

where σ_{rr}, $\sigma_{\theta\theta}$, and σ_{xx} are the normal stresses in the r, θ, and x directions, respectively, σ_{rx} is the shear stress in the r, x, plane, ρ is the mass density of the propellant, and u_r and u_x are the displacements in the r and x directions measured with respect to the moving coordinate system. Identical equations hold for the motor case with ρ denoting the mass density of the case.

The axisymmetric strain-displacement relationships are

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r} \tag{2a}$$

$$\epsilon_{\theta\theta} = \frac{u_r}{r} \tag{2b}$$

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x} \tag{2c}$$

$$\epsilon_{rx} = \frac{\partial u_r}{\partial x} + \frac{\partial u_x}{\partial r} \tag{2b}$$

The elastic constituitive relations for both the case and the propellant can be given in the form*

$$\sigma_{rr} = \lambda\Delta + 2\mu\epsilon_{rr} \tag{3a}$$

$$\sigma_{\theta\theta} = \lambda\Delta + 2\mu\epsilon_{\theta\theta} \tag{3b}$$

---

*Propellants are also viscous. The assumption is made here that the loading rate is sufficiently slow so that the viscous behavior can be neglected.

$$\sigma_{xx} = \lambda\Delta + 2\mu\epsilon_{xx} \tag{3c}$$

$$\sigma_{rx} = \mu\epsilon_{rx} \tag{3d}$$

where λ and μ are Lame's constants and Δ is the dilation given by

$$\Delta = \epsilon_{rr} + \epsilon_{\theta\theta} + \epsilon_{xx} \tag{3e}$$

For the incompressible propellant material, $\lambda = \infty$ and $\Delta = 0$.

NATURAL FREQUENCIES

As stated in the introduction, the objective of this effort is to find the lowest natural frequency of both the unbonded propellant and the bonded propellant-motor case system. These natural frequencies can be obtained by solving Eqs. (1) - (3) with the term $\frac{\partial^2 \bar{U}}{\partial t^2}$ set equal to zero*. The specific values for the frequencies will depend upon the boundary conditions of system and its physical and geometrical properties.

Armenàkas has considered the problem of the natural frequencies of bonded composite (two material) circular cylinders in Ref. 3. He assumed the displacement of both the propellant and the case could be expressed in the form

$$u_r = F(r) \cos \xi x \, e^{i\omega t} \tag{4a}$$

$$u_x = G(r) \sin \xi x \, e^{i\omega t} \tag{4b}$$

---

* This term is the "loading" term. It's this term that causes the elastic stresses and sets the propellant and case in motion relative to the rigid body motion of the rocket during launch. It does not enter into the equations when natural frequencies are sought.

where $F(r)$ and $G(r)$ are arbitrary functions of r, ξ is the longitudinal wave number and ω is the natural circular frequency. For the situation considered in this report, take

$$\xi = n\pi/L \tag{5}$$

where L is the length of the propellant, and $n = 0, 1, 2 \ldots \infty$. This leads to

$$\left.\begin{array}{l} u_x = 0 \; ; \; u_r \neq 0 \\ \\ \sigma_{rx} = 0 \; ; \; \sigma_{rr}, \sigma_{\theta\theta}, \sigma_{xx} \neq 0 \end{array}\right\} x = 0, L \tag{6}$$

according to Eqs.(2-5). The same conditions given by Eq. (6) apply to the motor case at the bulkheads. Thus, at the fore and aft bulkheads the propellant can slip along the bulkheads in the radial direction, but is restrained from moving in the direction normal to the bulkheads. This assumption on the displacement state is not entirely accurate since an unbonded propellant can pull away from the bulkhead, i.e. $u_x \neq 0$, and a bonded propellant would not slip along the bulkhead, i.e. $\sigma_{rx} \neq 0$. Furthermore, the net force in the axial direction should be zero at each bulkhead because the case is idealized as being in a free-free condition at the ends. This condition cannot be imposed. Nevertheless, in spite of these inadequacies, the analysis by Armenàkas will be used because it provides an estimate of the lowest natural frequency of the propellant and the propellant-case system.

The natural frequencies of the composite cylinder depend upon the conditions at the inner and outer surfaces of the propellant and the case. For the bonded propellant the inner surface of the propellant is stress

free, the propellant is perfectly bonded to the case at the interface, and the outer surface of the case is stress free. Hence,

$$
\begin{aligned}
&\text{at} \quad r = a: \quad \sigma_{rr_1} = \sigma_{rx_1} = 0 \\
&\text{at} \quad r = b: \quad \sigma_{rr_1} = \sigma_{rr_2}, \qquad u_{r_1} = u_{r_2} \\
&\text{and} \qquad\qquad \sigma_{rx_1} = \sigma_{rx_2}, \qquad u_{x1} = u_{x2} \\
&\text{at} \quad r = d: \quad \sigma_{rr_2} = \sigma_{rx_2} = 0
\end{aligned}
\tag{7}
$$

where a, b, and d are radii indicated in Fig. 1, and the subscripts 1 and 2 refer to the propellant and the case respectively.

For the unbonded propellant two conditions are considered. In the first, the case is assumed to be rigid and $u_{r_1} = \sigma_{rx_1} = 0$ at $r = b$. In the second, the case is neglected, and $\sigma_{rr_1} = \sigma_{rx_1} = 0$ at $r = b$.

The analysis by Armenàkas leads to an 8 x 8 matrix. The values for the natural frequency ω are obtained by equating the determinant of this matrix to zero. Each row in the matrix corresponds to one of the conditions of Eqs. (7). The coefficients of the matrix, which are complicated functions of Bessel functions, are given in Appendix A. The natural frequency is contained in the arguments of the Bessel functions. For a detailed presentation of the analysis refer to Ref. 3. For the unbonded propellant, the 8 x 8 matrix for the two material cylinder reduces to a 4 x 4 matrix for the one material cylinder (propellant) that is a special form of Armenàkas' 8 x 8 matrix. The elements of the 4 x 4 matrix for both the restrained propellant and the free propellant are also given in Appendix A.

A digital computer program was written to compute the determinant of the 8 x 8 matrix and the two 4 x 4 matrices for an assumed value of ω . The assumed value of ω was started at 300 rad/sec and incremented by 20 rad/sec in a Do loop. The lowest natural frequency was obtained by observing when the determinant first changed sign. The value for the natural frequency was refined by using a linear approximation between the adjacent + and - determinants to obtain the point when the determinant was zero. A program listing is given in Appendix B.

Three Inch Rocket

The following properties and dimensions were assumed for the three inch rocket:

$L = 9.5$ in.

$a = 0.50$ in.

$b = 1.42$ in.

$d = 1.50$ in.

Case

$$\rho = 7.33 \times 10^{-4} \frac{\#\text{-sec}^2}{\text{in.}^4}$$

$$\left.\begin{array}{l} \lambda = 19.5 \times 10^6 \text{ psi} \\ \mu = 11.0 \times 10^6 \text{ psi} \end{array}\right\} \quad E = 29 \times 10^6 \text{ psi}, \ \nu = 0.32$$

Propellant

$$\rho = 1.55 \times 10^{-4} \frac{\#\text{-sec}^2}{\text{in.}^4}$$

$$\left.\begin{array}{l} \lambda \approx \infty \text{ psi} \\ \mu = 300,\ 500,\ 700 \text{ psi} \end{array}\right\} \quad \begin{array}{l} E = 900,\ 1500,\ 2100 \text{ psi} \\ \nu \approx 0.5 \end{array}$$

where E is Young's modulus and ν is Poisson's ratio*.

*A study of the effect of Poisson's ratio upon the natural frequencies revealed that no difficulties were encountered as $\nu \rightarrow 0.5$. This is not the case in an elastic analysis. There, as $\nu \rightarrow 0.5$ numerical problems enter in and a solution with $\nu = 0.5$ is not possible. This feature does not occur with the natural frequency analysis. Thus, it was possible to examine an incompressible propellant since there was very little change in the lowest natural frequency for the range $0.45 \leq \nu \leq 0.50$.

The lowest natural frequency of the three inch motor is given in Table I for the three interface conditions with E (propellant) = 900, 1500 and 2100 psi.

For comparison purposes, the lowest natural frequency of each material, treated as a one-dimensional bar, is $\frac{\pi}{L}\sqrt{\frac{E}{\rho}}$. Thus,

$$\omega_{low,\ bar,\ case} = 6.54 \times 10^4 \text{ rad/sec} \quad (10{,}220 \text{ Hz})$$

$$\omega_{low,\ bar,\ prop} = 800,\ 1030,\ 1220 \text{ rad/sec} \quad (127,\ 164,\ 194 \text{ Hz})$$

The three values of ω for the propellant correspond to the three values assumed for E .

<u>Five Inch Rocket</u>

The following dimensions were assumed for the five inch rocket.

L = 20.0 in.

a = 0.5 in.

b = 2.32 in.

d = 2.5 in.

The properties of the propellant and case materials are the same as in the three inch rocket.

The lowest natural frequency of the five inch motor is given in Table II for the three interface conditions with E (propellant) = 900, 1500 and 2100 psi.

The lowest natural frequency of each material, treated as a one-dimensional bar, is

$$\omega_{low,\ bar,\ case} = 3.10 \times 10^4 \text{ rad/sec} \quad (4,\ 940 \text{ Hz})$$

$$\omega_{low,\ bar,\ prop} = 380,\ 490,\ 580 \text{ rad/sec} \quad (60.5,\ 78.0,\ 92.4 \text{ Hz})$$

Note that for both rockets the approximate natural frequency obtained by treating the propellant as a one-dimensional bar is very close to the natural frequency from the unbonded, no case two-dimensional analysis. This is due to the long, thin shape of the rocket.

DISCUSSION AND CONCLUSIONS

The object of this effort was to determine the lowest natural frequency of the propellant and to compare it with the lowest forcing frequency $\frac{\pi}{\alpha}$. Examining the results given in Tables I and II reveals that the lowest natural frequency for the unbonded propellant in a rigid case is in the neighborhood of 238 - 362 Hz for the three inch rocket and 167 - 254 Hz for the five inch rocket. The lowest forcing frequency is $\frac{\pi}{0.012} = 262$ rad/sec (Ref. 1) or ≈ 42 Hz for the three inch rocket and $\frac{\pi}{0.026} = 121$ rad/sec (Ref. 2) or ≈ 19 Hz for the five inch rocket. Thus, the frequency ratio is ≈ 8 . The frequency ratio for the case is $10{,}220/42 = 240$ and $4{,}940/19 = 260$ for the three and five inch rockets respectively. The dynamic response results presented in Ref. 1 showed that the dynamic effects due to the lowest forcing frequency were negligible in the steel case due to the frequency ratio of 240. However, when the frequency ratio was lowered to 76 by reducing E by a factor of 10, the dynamic effects became noticeable. Thus, it appears that the low frequency ratio of 8 for the propellant will lead to significant dynamic effects. Furthermore, the higher harmonics in the forcing function will cause additional dynamic effects. The significance of these effects depends upon the magnitude of the higher harmonics. A very sharp-fronted pressure pulse will cause very large dynamic effects in the propellant. This is demonstrated in Ref. 2 where a complex pressure pulse is considered. Reducing the frequency ratio from 255 to 51 caused dynamic stresses nearly twice those due to mass inertia alone.

Bonding the propellant to the case stiffens the propellant and raises the frequency ratio, thus reducing the dynamic effects somewhat, but not enough to eliminate the problem.

Condition \ E, psi	900	1500	2100
propellant bonded to the case	3173 rad/sec 505 Hz	4088 rad/sec 650 Hz	4829 rad/sec 768 Hz
unbonded propellant, rigid case	1491 rad/sec 238 Hz	1923 rad/sec 306 Hz	2278 rad/sec 362 Hz
unbonded propellant, no case	784 rad/sec 125 Hz	1012 rad/sec 161 Hz	1198 rad/sec 191 Hz

Table I

Lowest Natural Frequency for the Three Inch Rocket

Condition \ E, psi	900	1500	2100
propellant bonded to the case	1805 rad/sec 288 Hz	2326 rad/sec 370 Hz	2743 rad/sec 436 Hz
unbonded propellant, rigid case	1048 rad/sec 167 Hz	1353 rad/sec 216 Hz	1601 rad/sec 254 Hz
unbonded propellant, no case	374 rad/sec 59.5 Hz	482 rad/sec 76.7 Hz	572 rad/sec 91.0 Hz

Table II

Lowest Natural Frequency for the Five Inch Rocket

REFERENCES

1. Ball, R. E. and Salinas, David, "Analysis of a Three Inch Gun Launched Finned Motor Case," Naval Postgraduate School, NPS-57Bp72011A, January 1972.

2. Salinas, David and Ball, R. E., "Transient Axial Response of a Gun Launched Rocket Motor Case During Launch," Naval Postgraduate School, NPS-57Zc73011A, January 29, 1973.

3. Armenàkas, A. E., "Propagation of Harmonic Waves in Composite Circular Cylindrical Shells. I: Theoretical Investigation," AIAA Journal, Vol. 5, No. 4, April 1967, pp. 740-744.

APPENDIX A--COEFFICIENTS OF THE MATRIX

For axisymmetric motion, the natural frequencies of the two material motor are obtained when the determinant of an 8 x 8 matrix is zero. The matrix, as derived by Armenakas (Ref. 3), is

$$
\begin{array}{cccccccccl}
C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 & 0 & 0 & (1 \\
C_{31} & C_{32} & C_{33} & C_{34} & 0 & 0 & 0 & 0 & (2 \\
0 & 0 & 0 & 0 & C_{47} & C_{48} & C_{49} & C_{4,10} & (3 \\
0 & 0 & 0 & 0 & C_{67} & C_{68} & C_{69} & C_{6,10} & (4 \\
C_{71} & C_{72} & C_{73} & C_{74} & C_{77} & C_{78} & C_{79} & C_{7,10} & (5 \\
C_{91} & C_{92} & C_{93} & C_{94} & C_{97} & C_{98} & C_{99} & C_{9,10} & (6 \\
C_{10,1} & C_{10,2} & C_{10,3} & C_{10,4} & C_{10,7} & C_{10,8} & C_{10,9} & C_{10,10} & (7 \\
C_{12,1} & C_{12,2} & C_{12,3} & C_{12,4} & C_{12,7} & C_{12,8} & C_{12,9} & C_{12,10} & (8
\end{array}
$$

where the coefficients are given by

$$C_{11} = 2K_{11}(a\bar{\alpha}_1)Z_1(a\bar{\alpha}_1) + [-(a\bar{\beta}_1)^2 + (a\xi)^2]Z_0(a\bar{\alpha}_1)$$

$$C_{12} = 2(a\bar{\alpha}_1)W_1(a\bar{\alpha}_1) + [-(a\bar{\beta}_1)^2 + (a\xi)^2]W_0(a\bar{\alpha}_1)$$

$$C_{13} = 2(a\xi)[(a\bar{\beta}_1)Z_0(a\bar{\beta}_1) - Z_1(a\bar{\beta}_1)]$$

$$C_{14} = 2(a\xi)[K_{21}(a\bar{\beta}_1)W_0(a\bar{\beta}_1) - W_1(a\bar{\beta}_1)]$$

$$C_{31} = 2(a\xi)[K_{11}(a\bar{\alpha}_1)Z_1(a\bar{\alpha}_1)]$$

$$C_{32} = 2(a\xi)[(a\bar{\alpha}_1)W_1(a\bar{\alpha}_1)]$$

$$C_{33} = -\{[(a\xi)^2 - (a\bar{\beta}_1)^2]Z_1(a\bar{\beta}_1)\}$$

$$C_{34} = -\{[(a\xi)^2 - (a\bar{\beta}_1)^2]W_1(a\bar{\beta}_1)\}$$

$$C_{47} = 2K_{12}(d\bar{\alpha}_2)Z_1(d\bar{\alpha}_2) + [(d\bar{\beta}_2)^2 + (d\xi)^2]Z_0(d\bar{\alpha}_2)$$

$$C_{48} = 2(d\bar{\alpha}_2)W_1(d\bar{\alpha}_2) + [(d\bar{\beta}_2)^2 + (d\xi)^2]W_0(d\bar{\alpha}_2)$$

$$C_{49} = 2(d\xi)[(d\bar{\beta}_2)Z_0(d\bar{\beta}_2) - Z_1(d\bar{\beta}_2)]$$

$$C_{4,10} = 2(d\xi)[K_{22}(d\bar{\beta}_2)W_0(d\bar{\beta}_2) - W_1(d\bar{\beta}_2)]$$

$$C_{67} = 2(d\xi)[K_{12}(d\bar{\alpha}_2)Z_1(d\bar{\alpha}_2)]$$

$$C_{68} = 2(d\xi)[(d\bar{\alpha}_2)W_1(d\bar{\alpha}_2)]$$

$$C_{69} = -\{[(d\xi)^2 - (d\bar{\beta}_2)^2]Z_1(d\bar{\beta}_2)\}$$

$$C_{6,10} = -\{[(d\xi)^2 - (d\bar{\beta}_2)^2]W_1(d\bar{\beta}_2)\}$$

$$c_{71} = \mu\{2K_{11}(b\bar{\alpha}_1)Z_1(b\bar{\alpha}_1) + [-(b\bar{\beta}_1)^2 + (b\xi)^2]Z_0(b\bar{\alpha}_1)\}$$

$$c_{72} = \mu\{2(b\bar{\alpha}_1)W_1(b\bar{\alpha}_1) + [-(b\bar{\beta}_1)^2 + (b\xi)^2]W_0(b\bar{\alpha}_1)\}$$

$$c_{73} = \mu\{2(b\xi)[(b\bar{\beta}_1)Z_0(b\bar{\beta}_1) - Z_1(b\bar{\beta}_1)]\}$$

$$c_{74} = \mu 2(b\xi)[K_{21}(b\bar{\beta}_1)W_0(b\bar{\beta}_1) - W_1(b\bar{\beta}_1)]$$

$$c_{77} = -\{2K_{12}(b\bar{\alpha}_2)Z_1(b\bar{\alpha}_2) + [(b\bar{\beta}_2)^2 + (b\xi)^2]Z_0(b\bar{\alpha}_2)\}$$

$$c_{78} = -\{2(b\bar{\alpha}_2)W_1(b\bar{\alpha}_2) + [(b\bar{\beta}_2)^2 + (b\xi)^2]W_0(b\bar{\alpha}_2)\}$$

$$c_{79} = -\{2(b\xi)[(b\bar{\beta}_2)Z_0(b\bar{\beta}_2) - Z_1(b\bar{\beta}_2)]\}$$

$$c_{7,10} = -\{2(b\xi)[K_{22}(b\bar{\beta}_2)W_0(b\bar{\beta}_2) - W_1(b\bar{\beta}_2)]\}$$

$$c_{91} = \mu\{2(b\xi)[K_{11}(b\bar{\alpha}_1)Z_1(b\bar{\alpha}_1)]\}$$

$$c_{92} = \mu\{2(b\xi)[(b\bar{\alpha}_1)W_1(b\bar{\alpha}_1)]\}$$

$$c_{93} = -\mu\{[(b\xi)^2 - (b\bar{\beta}_1)^2]Z_1(b\bar{\beta}_1)\}$$

$$c_{94} = -\mu\{[(b\xi)^2 - (b\bar{\beta}_1)^2]W_1(b\bar{\beta}_1)\}$$

$$c_{97} = -\{2(b\xi)[K_{12}(b\bar{\alpha}_2)Z_1(b\bar{\alpha}_2)]\}$$

$$c_{98} = -\{2(b\xi)[(b\bar{\alpha}_2)W_1(b\bar{\alpha}_2)]\}$$

$$c_{99} = +\{[(b\xi)^2 - (b\bar{\beta}_2)^2]Z_1(b\bar{\beta}_2)\}$$

$$c_{9,10} = +\{[(b\xi)^2 - (b\bar{\beta}_2)^2]W_1(b\bar{\beta}_2)\}$$

$$C_{10,1} = - K_{11}(b\bar{\alpha}_1)\, Z_1(b\bar{\alpha}_1)$$

$$C_{10,2} = - (b\bar{\alpha}_1)\, W_1(b\bar{\alpha}_1)$$

$$C_{10,3} = (b\xi)\, Z_1(b\bar{\beta}_1)$$

$$C_{10,4} = (b\xi)\, W_1(b\bar{\beta}_1)$$

$$C_{10,7} = + K_{12}(b\bar{\alpha}_2)\, Z_1\, (b\bar{\alpha}_2)$$

$$C_{10,8} = + (b\bar{\alpha}_2)\, W_1(b\bar{\alpha}_2)$$

$$C_{10,9} = - (b\xi)\, Z_1(b\bar{\beta}_2)$$

$$C_{10,10} = - (b\xi)\, W_1(b\bar{\beta}_2)$$

$$C_{12,1} = - (b\xi)\, Z_0(b\bar{\alpha}_1)$$

$$C_{12,2} = - (b\xi)\, W_0(b\bar{\alpha}_1)$$

$$C_{12,3} = - (b\bar{\beta}_1)\, Z_0\, (b\bar{\beta}_1)$$

$$C_{12,4} = - K_{21}(b\bar{\beta}_1)\, W_0(b\bar{\beta}_1)$$

$$C_{12,7} = + (b\xi)\, Z_0(b\bar{\alpha}_2)$$

$$C_{12,8} = + (b\xi)\, W_0(b\bar{\alpha}_2)$$

$$C_{12,9} = + (b\bar{\beta}_2)\, Z_0(b\bar{\beta}_2)$$

$$C_{12,10} = + K_{22}(b\bar{\beta}_2)\, W_0(b\bar{\beta}_2)$$

where

$$\mu = \mu_1/\mu_2$$

$$\bar{\alpha}_1 = \left| \frac{\omega^2}{v_{1,i}^2} - \xi^2 \right| \quad , \quad \bar{\beta}_i = \left| \frac{\omega^2}{v_{2,i}^2} - \xi^2 \right|$$

$$v_{1,i}^2 = \frac{\lambda_i + 2\mu_i{}^*}{\rho_i} \quad , \quad v_{2,i}^2 = \frac{\mu_i}{\rho_i}$$

The subscript i denotes the propellant $(i = 1)$ and the case $(i = 2)$.
Z_n and W_n denote a Bessel function J, Y, I or K according to the following scheme:

	Z_n	W_n	Z_n	W_n
$\omega > \xi\, v_{1,i}$	$J_n(\bar{\alpha}_i r)$	$Y_n(\bar{\alpha}_i r)$	$J_n(\bar{\beta}_i r)$	$Y_n(\bar{\beta}_i r)$
$\xi v_{2,i} < \omega < \xi v_{1,i}$	$I_n(\bar{\alpha}_i r)$	$K_n(\bar{\alpha}_i r)$	$J_m(\bar{\beta}_i r)$	$Y_n(\bar{\beta}_i r)$
$\omega < \xi v_{2,i}$	$I_n(\bar{\alpha}_i r)$	$K_n(\bar{\alpha}_i r)$	$I_n(\bar{\beta}_i r)$	$K_n(\bar{\beta}_i r)$

$$K_{ij} = -1 \quad \text{if} \quad \omega < \xi v_{i,j}$$

$$K_{ij} = 1 \quad \text{if} \quad \omega > \xi v_{i,j}$$

---

*Note that when $\lambda \to \infty$, $v_{1,i} \to \infty$ and $\bar{\alpha}_i \to |-\xi^2|$. Thus, there appear to be no difficulties with an incompressible material. This was verified computationally by letting $\nu \to 0.5$ and comparing the lowest natural frequency computed for each value of ν .

The first two rows (1 and 2) in the 8 x 8 matrix are the stress free boundary conditions on the propellant at its inner surface. The next two rows (3 and 4) are the stress free boundary conditions on the case at its outer surface. The last four rows are the continuity conditions on σ_{rr} (5), σ_{rx} (6), u_{rr} (7) and u_{rx} (8) respectively. For the natural frequencies of the unbonded propellant in a rigid case, the 8 x 8 reduces to a 4 x 4 by considering only rows 1, 2, 6, and 7, of the 8 x 8, i.e., $\sigma_{rx} = u_{rr} = 0$ at $r = b$. Thus

$$
\begin{matrix}
C_{11} & C_{12} & C_{13} & C_{14} \\
C_{31} & C_{32} & C_{33} & C_{34} \\
C_{91} & C_{92} & C_{93} & C_{94} \\
C_{10,1} & C_{10,2} & C_{10,3} & C_{10,4}
\end{matrix}
$$

is the governing matrix. For the unbonded propellant with no case, rows 1, 2, 5, and 6, are retained, i.e. $\sigma_{rr} = \sigma_{rx} = 0$ at $r = b$. Thus,

$$
\begin{matrix}
C_{11} & C_{12} & C_{13} & C_{14} \\
C_{31} & C_{32} & C_{33} & C_{34} \\
C_{71} & C_{72} & C_{73} & C_{74} \\
C_{91} & C_{92} & C_{93} & C_{94}
\end{matrix}
$$

is the governing matrix.

```
C    NATURAL FREQUENCIES OF COMPOSITE CYLINDERS
//    EXEC FORTCLG,REGION=150K
//FORT.SYSIN DD DSNAME=SSP3(BESJ),DISP=SHR
//           DD DSNAME=SSP3(BESY),DISP=SHR
//           DD DSNAME=SSP3(INUE),DISP=SHR
//           DD DSNAME=SSP3(BESK),DISP=SHR
//           DD DSNAME=SSP3(IO),DISP=SHR
//           DD *
      DIMENSION C(8,8),L(4),N(4),ARGA(4),ARGB(4),AIO(4),AI1(4),AKO(4)
      DIMENSION AK1(4),AJO(4),AJ1(4),AYO(4),AY1(4)
      DIMENSION C1(4,4),C2(4,4),
     1  BIO(4),BI1(4),BKO(4),BK1(4)
     2,BJO(4),BJ1(4),BYO(4),BY1(4)
C   POISSON'S RATIO = ENU
      ENU=.5
      EN=1.
C    LENGTH=20. INCHES
      XI=EN*3.14159265/20.
      YM=300.
      DO 998 LL=1,3
      YM=YM+600.
      V21S=(    YM/(2.*(1.+ENU)))/1.55E-04
      IF(ENU.GE.0.5) GO TO 22
      V11S=(ENU*    YM/((1.+ENU)*(1.-2.*ENU)))/1.55E-04+2.*V21S
   22 CONTINUE
      V12S=(.32*29.E+06/(1.32*.36)+22.E+06)/(.283/386.)
      V22S=11.E+06/(.283/386.)
      XIS=XI*XI
      AMU=(    YM/(2.*(1.+ENU)))/11.E+06
      WRITE(6,9)
    9 FORMAT(1H1,10X,'OMEGA',10X,'TWO MATERIALS',10X,'PROPELLANT,RIGID C
     1ASE',10X,'PROPELLANT, NO CASE'///)
      DO 1 I=1,360
      A1=XIS
      OMEG=300.+FLOAT(I-1)*20.
      OMEGS=OMEG*OMEG
      IF(ENU.GE.0.5) GO TO 23
      A1=ABS(OMEGS/V11S-XIS)
   23 CONTINUE
      B1=ABS(OMEGS/V21S-XIS)
      A2=ABS(OMEGS/V12S-XIS)
      B2=ABS(OMEGS/V22S-XIS)
      A1=SQRT(A1)
      B1=SQRT(B1)
      A2=SQRT(A2)
      B2=SQRT(B2)
C    INNER RADIUS=0.5 INCHES
      AR=0.5
```

```
C    INTERFACE RADIUS=2.31 INCHES
        BR=2.31
C    CUTER RADIUS=2.50 INCHES
        DR=2.50
        PI=3.14159265
        ARGA(1)=AR*A1
        ARGA(2)=BR*A1
        ARGA(3)=BR*A2
        ARGA(4)=DR*A2
        ARGB(1)=AR*B1
        ARGB(2)=BR*B1
        ARGB(3)=BR*B2
        ARGB(4)=DR*B2
        D=1.E-5
        DO 2 K=1,4
        CALL IO(ARGA(K),AIO(K))
        CALL IO(ARGB(K),BIO(K))
        CALL INUE(ARGA(K),1,AIO(K),AI1(K))
        CALL INUE(ARGB(K),1,BIO(K),BI1(K))
        CALL BESK(ARGA(K),0,AKO(K),IER)
        CALL BESK(ARGB(K),0,BKO(K),IER)
        CALL BESK(ARGA(K),1,AK1(K),IER)
        CALL BESK(ARGB(K),1,BK1(K),IER)
        CALL BESJ(ARGA(K),0,BJO(K),D,IER)
        CALL BESJ(ARGB(K),0 ,AJO(K),D,IER)
        CALL BESJ(ARGA(K),1,BJ1(K),D,IER)
        CALL BESJ(ARGB(K),1,AJ1(K),D,IER)
        CALL BESY(ARGA(K),0,BYO(K),IER)
        CALL BESY(ARGB(K),0,AYO(K),IER)
        CALL BESY(ARGA(K),1,BY1(K),IER)
        CALL BESY(ARGB(K),1,AY1(K),IER)
      2 CONTINUE
        IF(OMEG.GT.19.15E+03) GO TO 80
     81 DO 82 K=3,4
        AJO(K)=BIO(K)
        AJ1(K)=BI1(K)
        AYO(K)=-BKO(K)
     82 AY1(K)=BK1(K)
     80 CONTINUE
        GO TO 27
     28 CONTINUE
C    THE FOLLOWING 6 CARDS ARE FOR ENU LESS THAN 0.5
        IF(OMEG.LT.6.4E+03) GO TO 60
        DC 61 KT=1,2
        AIO(KT)=BJO(KT)
        AI1(KT)=-BJ1(KT)
        AKO(KT)=BYO(KT)
     61 AK1(KT)=BY1(KT)
```

```
60 CONTINUE
27 CONTINUE
   DO 3 KK=1,8
   DO 3 JJ=1,8
 3 C(JJ,KK)=0.
   C(1,1)=-2.*ARGA(1)*AI1(1)+(-ARGB(1)*ARGB(1)+AR*AR*XIS)*AIO(1)
   C(1,2)= 2.*ARGA(1)*AK1(1)+(-ARGB(1)*ARGB(1)+AR*AR*XIS)*AKO(1)
   C(1,3)= 2.*AR*XI*(ARGB(1)*AJO(1)-AJ1(1))
   C(1,4)= 2.*AR*XI*(ARGB(1)*AYO(1)-AY1(1))
   C(2,1)=2.*AR*XI*     ARGA(1)*AI1(1)*(-1.)
   C(2,2)=2.*AR*XI*ARGA(1)*AK1(1)
   C(2,3)=-(AR*AR*XIS-ARGB(1)*ARGB(1))*AJ1(1)
   C(2,4)=-(AR*AR*XIS-ARGB(1)*ARGB(1))*AY1(1)
   C(3,5)=-2.*ARGA(4)*AI1(4)+(-ARGB(4)*ARGB(4)+DR*DR*XIS)*AIO(4)
   C(3,6)= 2.*ARGA(4)*AK1(4)+(-ARGB(4)*ARGB(4)+DR*DR*XIS)*AKO(4)
   C(3,7)= 2.*DR*XI*(ARGB(4)*AJO(4)-AJ1(4))
   C(3,8)= 2.*DR*XI*(ARGB(4)*AYO(4)-AY1(4))
   C(4,5)= 2.*DR*XI*     ARGA(4)*AI1(4)*(-1.)
   C(4,6)= 2.*DR*XI*ARGA(4)*AK1(4)
   C(4,7)= -(DR*DR*XIS-ARGB(4)*ARGB(4))*AJ1(4)
   C(4,8)= -(DR*DR*XIS-ARGB(4)*ARGB(4))*AY1(4)
   C(5,1)=AMU*(-2.*ARGA(2)*AI1(2)+(-ARGB(2)*ARGB(2)+BR*BR*XIS)*AIO(2)
  1)
   C(5,2)=AMU*(2.*ARGA(2)*AK1(2)+(-ARGB(2)*ARGB(2)+BR*BR*XIS)*AKO(2))
   C(5,3)=AMU*2.*BR*XI*(ARGB(2)*AJO(2)-AJ1(2))
   C(5,4)=AMU*2.*BR*XI*(ARGB(2)*AYO(2)-AY1(2))
   C(5,5)=-(-2.*ARGA(3)*AI1(3)+(-ARGB(3)*ARGB(3)+BR*BR*XIS)*AIO(3))
   C(5,6)=-( 2.*ARGA(3)*AK1(3)+(-ARGB(3)*ARGB(3)+BR*BR*XIS)*AKO(3))
   C(5,7)=-( 2.*BR*XI*(ARGB(3)*AJO(3)-AJ1(3)))
   C(5,8)=-( 2.*BR*XI*(ARGB(3)*AYO(3)-AY1(3)))
   C(6,1)=AMU*2.*BR*XI*     ARGA(2)*AI1(2)*(-1.)
   C(6,2)=AMU*2.*BR*XI*ARGA(2)*AK1(2)
   C(6,3)= -AMU*(BR*BR*XIS-ARGB(2)*ARGB(2))*AJ1(2)
   C(6,4)=-AMU*(BR*BR*XIS-ARGB(2)*ARGB(2))*AY1(2)
   C(6,5)=-2.*BR*XI*     ARGA(3)*AI1(3)*(-1.)
   C(6,6)=-2.*BR*XI*ARGA(3)*AK1(3)
   C(6,7)= (BR*BR*XIS-ARGB(3)*ARGB(3))*AJ1(3)
   C(6,8) = (BR*BR*XIS-ARGB(3)*ARGB(3))*AY1(3)
   C(7,1)= ARGA(2)*AI1(2)
   C(7,2)= -ARGA(2)*AK1(2)
   C(7,3)= BR*XI*AJ1(2)
   C(7,4)= BR*XI*AY1(2)
   C(7,5)= -ARGA(3) *AI1(3)
   C(7,6)= ARGA(3)*AK1(3)
   C(7,7)= -BR*XI*AJ1(3)
   C(7,8)= -BR*XI*AY1(3)
   C(8,1)= -BR*XI*AIO(2)
   C(8,2)= -BR*XI*AKO(2)
```

```
      C(8,3)= -ARGB(2)*AJO(2)
      C(8,4)= -ARGB(2)*AYO(2)
      C(8,5)=  BR*XI*AIO(3)
      C(8,6)=  BR*XI*AKO(3)
      C(8,7)=  ARGB(3)*AJO(3)
      C(8,8)=  ARGB(3)*AYO(3)
C    C1 IS THE MATRIX FOR THE UNBONDED PROPELLANT WITH NO CASE
C    C2 IS THE MATRIX FOR THE UNBONDED PROPELLANT IN A RIGID CASE
      DO 31 KK=1,2
      DO 31 JJ=1,4
      C2(KK,JJ)=C(KK,JJ)
   31 C1(KK,JJ)=C(KK,JJ)
      DO 32 KK=3,4
      DO 32 JJ=1,4
      C2(3,JJ)=C(6,JJ)
      C2(4,JJ)=C(7,JJ)
   32 C1(KK,JJ)=C(KK+2,JJ)
      CALL MINV(C,8,DETC,L,N)
      CALL MINV(C1,4,DETC1,L,N)
      CALL MINV(C2,4,DETC2,L,N)
      WRITE(6,10) OMEG,DETC,DETC2,DETC1
   10 FORMAT(' ',10X,F5.0,10X,E13.4,10X,E21.4,10X,E19.4)
    1 CONTINUE
  998 CONTINUE
      STOP
      END
```

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